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# $C^1$ SELF–MAPS ON CLOSED MANIFOLDS WITH ALL THEIR PERIODIC POINTS HYPERBOLIC

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ABSTRACT. We present several results providing sufficient conditions for the existence of almost quasi–unipotent maps on different closed manifolds having infinitely many periodic points all of them hyperbolic.

## 1. Introduction and definitions

Let $X$ be a topological space and $f : X \to X$ be a continuous map on $X$. We say that $x \in X$ is a *periodic point of period* $p$ if $f^p(x) = x$ and $f^j(x) \neq x$ if $1 \leq j \leq p - 1$.

Let $X$ be a differentiable manifold and $f$ a differentiable map. We say that a periodic point of period $p$ is *hyperbolic*, if the derivative of $f^p$ at $x$, i.e. $Df^p_x : TX_x \to TX_x$, has no eigenvalues of modulus equal 1.

We say that a manifold is *closed* if it is compact and without boundary.

If the dimension of $X$ is $n$, the map $f$ induces a homomorphism on the $k$–th rational homological group of $X$ for $0 \leq k \leq n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and $f_{*k}$ is a linear map whose matrix has integer entries.

A linear transformation is called *quasi–unipotent* if its eigenvalues are roots of unity. We say that a continuous map $f : X \to X$ is *quasi–unipotent* if the maps $f_{*k}$ are quasi–unipotent, for $0 \leq k \leq n$, where $n$ is the dimension of the manifold $X$.

We remark that there diffeomorphisms with finitely many periodic points, all of them hyperbolic, which are quasi-unipotent like the Morse-Smale diffeomorphisms [8]. These maps are an important family of dynamical systems. On the other hand, in the classical construction of the Smale's horseshoe (*cf.* [9]); there is a diffeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$, such that it has infinitely many periodic points all of them hyperbolic and with

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all possible periods. The map $f$ is quasi–unipotent. There are maps on the $n$-dimensional torus, which are minimal (all orbits are dense) and they are quasi–unipotent (*cf.* [2]).

In section 2 we present six theorems giving sufficient conditions for the existence of almost quasi–unipotent maps on different closed manifolds having infinitely many periodic points all of them hyperbolic. These results generalise the conditions given J. Franks in [3]. We note that the results of section 2 can be extended to manifolds with boundary which do not have periodic points on the boundary.

The *Lefschetz number* of $f$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}).$$

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then $f$ has a fixed point (*cf.* [1]).

The *Lefschetz zeta function* of $f$ is defined as

$$\zeta_f(t) = \exp\left( \sum_{m\geq 1} \frac{L(f^m)}{m} t^m \right).$$

Since $\zeta_f(t)$ is the generating function of the Lefschetz numbers, $L(f^m)$, it keeps the information of the Lefschetz number for all the iterates of $f$. There is an alternative way to compute it:

$$\zeta_f(t) = \prod_{k=0}^{n} \det(Id_k - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim X$, $m_k = \dim H_k(X, \mathbb{Q})$, $Id_k$ is the identity map on $H_k(X, \mathbb{Q})$, and by convention $\det(Id_k - t f_{*k}) = 1$ if $m_k = 0$ (*cf.* [4]).

Let $M$ be a $C^1$ compact manifold and let $f : M \to M$ be a $C^1$ map. Let $x$ be a hyperbolic periodic point of period $p$ of $f$ and $E^u_x$ its unstable linear space, i.e. the subspace of the tangent space $T_x M$ generated by the eigenvalues of $Df^p(x)$ of norm larger than 1. Let $\gamma$ be the orbit of $x$, the *index* $u$ of $\gamma$ is the dimension of $E^u_x$. We define the orientation type $\Delta$ of $\gamma$ as $+1$ if $Df^p(x) : E^u_x \to E^u_x$ preserves orientation and $-1$ if reverses the orientation. The collection of the triples $(p, u, \Delta)$ belonging to all the periodic orbits of $f$ is called the *periodic data* of $f$. The same triple can appear more than once if it corresponds to different periodic orbits.

**Theorem 1** (Franks [3])**.** *Let $f$ be a $C^1$ map on a closed manifold having finitely many periodic points all of them hyperbolic, and let $\Sigma$ be the* *periodic data of $f$. Then the Lefschetz zeta function $\zeta_f(t)$ of $f$ satisfies*

$$\zeta_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1-\Delta t^p)^{(-1)^{u+1}}. \tag{2}$$

## 2. $C^1$ MAPS WITH INFINITELY MANY HYPERBOLIC PERIODIC POINTS

As usual we shall consider that 1 is a power of 2.

**Theorem 2.** *Let $f$ be a $C^1$ map on $\mathbb{S}^{2n}$ having only hyperbolic periodic points.*

(a) *If $f$ is an orientation preserving map, having at most one periodic orbit with even index of period a power of 2 different from 1, then $f$ has infinitely many periodic points.*

(b) *If $f$ is either an orientation reversing map, or $f$ has degree 0, and it does not have periodic points with even index with period a power of 2, then $f$ has infinitely many periodic points.*

This theorem was proved in [3] when $n = 1$, similar arguments work for $n > 1$, but for sake of completeness we present the proof here.

We first point out a remark done in [3]. The polynomials of the form $1+t^m$ and $1-t^n$, with $m$ a positive integer and $n$ an odd positive integer, cannot be further factorized in the form

$$\prod_{i=1}^{l}(1 \pm t^{p_i}). \tag{3}$$

On the other hand $1 - t^{2m} = (1 - t^m)(1 + t^m)$.

A factorization of a polynomial of type (3) which cannot be further factorized and keeping the form (3) will be called a *special factorization.* The special factorization is unique up to sign and order of the factors, see Lemma 2 of [3].

*Proof of Theorem 2.* If $f$ is orientation preserving, then from (1) we get $\zeta_f(t) = 1/(1-t)^2$. Assume that $f$ under the assumptions of statement (a) has finitely many periodic points. Therefore, by Theorem 1, if $f$ has finitely many periodic points then

$$\zeta_f(t) = \frac{\prod_i (1 \pm t^{p_i})}{\prod_j (1 \pm t^{q_j})}.$$

So

$$\prod(1 \pm t^{q_j}) = (1-t)^2 \prod(1 \pm t^{p_i}). \tag{4}$$

Since there is at most one point with index even and its period is a power of 2 different from 1, there is only one $q_i$ which is a power of 2 different from 1. So, there is at most one factor of the form $1-t$ in the left hand side of (4) when the special factorization is taken. On the right hand side of (4), the factor $1-t$ appears with a power greater than or equal to 2. Therefore the equality of (4) does not hold, hence $f$ has infinitely many periodic points and all of them hyperbolic. This completes the proof of statement (a).

If $f$ is orientation reversing, then $\zeta_f(t) = ((1+t)(1-t))^{-1}$. If the map $f$ has finitely many hyperbolic periodic points, by Theorem 1 we have

$$\prod(1 \pm t^{q_j}) = (1-t)(1+t)\prod(1 \pm t^{p_i}). \tag{5}$$

We take the special factorization on booth sides of this equation. Since $f$ does not have periodic points of even index with period a power of 2. Then, there is no factors of the form $1-t$ on the left hand side of (5). On the right hand side of (5) there is at least one factor of this form. Therefore the equality of (5) does not hold, so $f$ has infinitely many periodic points. If the degree of $f$ is 0, then $\zeta_f(t) = (1-t)^{-1}$ and a similar argument allow us to conclude that $f$ has infinitely many periodic points . This completes the proof of statement (b). □

Similar arguments to the ones of the proof of Theorem 2 can be used for proving the following the Theorems 3, 4 and 5.

**Theorem 3.** *Let $X$ be a closed manifold and $f : X \to X$ be a $C^1$ map with all its periodic points hyperbolic. If $f$ has neither periodic points of even index with period 2, nor fixed points, and*

$$\zeta_f(t) = \frac{\prod_{i=1}^{l}(1 \pm t^{n_i})}{(1-t)^{m_1}(1+t)^{m_2}},$$

*where $m_1$ and $m_2$ are positive integers and the $n_i$'s are odd integers such that $n_i \geq 3$. Then $f$ has infinitely many periodic points.*

*Proof.* If $f$ has finitely many periodic points then, according to Theorem 1, we have

$$\prod_i(1 \pm t^{n_i})\prod_j(1 \pm t^{q_j}) = (1-t)^{m_1}(1+t)^{m_2}\prod_l(1 \pm t^{p_l}). \tag{6}$$

If $f$ does not have periodic points of even index of period 2, nor fixed points. Then there is no factor of the form $1-t$ in the left hang side of the (6), when the special factorization is taken. However in the right hand side the factor $1-t$ appears at least $m_1$ times. So it is not possible to have an equality in (6). Therefore $f$ has infinitely many periodic points. □

**Theorem 4.** *Let $X$ be a closed manifold and $f : X \to X$ be a $C^1$ map with all its periodic points hyperbolic. If $f$ has at most one periodic orbit of even index with period a power of 2 different from 1 and*

$$\zeta_f(t) = \frac{p(t)}{(1-t)^m},$$

*where $m \geq 2$ and $p(t)$ is a polynomial that could have one of the following forms*

(a) $p(t) = 1$,

(b) $p(t) = \prod_{i=1}^{l_1}(1 \pm t^{n_i})$, *where the $n_i$'s are odd integers greater than 2,*

(c) $p(t) = \prod_{j=1}^{l_2}(1 + t^{2^{k_j}})$, *where the $k_j$'s are positive integers,*

(d)
$$p(t) = \left(\prod_{i=1}^{l_1}(1 \pm t^{n_i})\right)\left(\prod_{j=1}^{l_2}(1 + t^{2^{k_j}})\right),$$
*where the $k_j$'s are positive integers and the $n_i$'s are odd integers greater than 2.*

*Then $f$ has infinitely many periodic points.*

*Proof.* We assume that $p(t)$ is as in (d). As in Theorem 3, if $f$ has finitely many periodic points then, according to Theorem 1, we have

$$\prod_i(1 \pm t^{n_i})\prod_j(1 + t^{2^{k_j}})\prod_s(1 \pm t^{q_s}) = (1-t)^m\prod_l(1 \pm t^{p_l}). \tag{7}$$

If $f$ has at most one point of even index then on the left hand side of (7) there is at most one factor of the form $(1-t)$, when the special factorization is taken. On the right hand side of (7) the factor $1-t$ appears at least with power $m$, which is greater than 2, so the equality (7) does not hold. Hence $f$ has infinitely many periodic points. This completes the proof of statement (d).

The proof of statements (a), (b) and (c) are similar. □

**Theorem 5.** *Let $X$ be a closed manifold and $f : X \to X$ be a $C^1$ map with all its periodic points hyperbolic. If $f$ has neither periodic points of even index with period 2, nor fixed points and*

$$\zeta_f(t) = \frac{p(t)}{(1-t)^{m_1}},$$

*where $p(t)$ is a polynomial that could have one of the following forms*

(a) $p(t) = 1$,

(b) $p(t) = \prod_{i=1}^{l_1}(1 \pm t^{n_i})$, *where the $n_i$'s are odd integers greater than 2,*

(c) $p(t) = \prod_{j=1}^{l_2}(1 + t^{2^{k_j}})$, *where the* $k_j$*'s are positive integers,*

(d)

$$p(t) = \left(\prod_{i=1}^{l_1}(1 \pm t^{n_i})\right)\left(\prod_{j=1}^{l_2}(1 + t^{2^{k_j}})\right),$$

*where the* $k_j$*'s are positive integers and the* $n_i$*'s are odd integers greater than* 2.

*Then* $f$ *has infinitely many periodic points.*

The proof of Theorem 5 is very similar to the proof of Theorem 4.

**Theorem 6.** *Let* $f$ *be a* $C^1$ *map on* $\mathbb{S}^{2n+1}$ *having only hyperbolic periodic points.*

(a) *If* $f$ *is an orientation preserving map having at least one periodic orbit of odd period* $p$ *with index* $u$ *and* $f$ *does not have periodic orbits of period even multiples of* $p$ *with index* $v \not\equiv u \pmod 2$*. Then* $f$ *has infinitely many periodic points.*

(b) *Assume that* $f$ *is an orientation preserving map which has periodic points of period a power of* 2 *whose indexes have the same parity. Let* $u$ *be this parity, if* $f$ *does not have fixed points of index* $v$ *with* $v \not\equiv u \pmod 2$*. Then* $f$ *has infinitely many periodic points.*

(c) *If* $f$ *either is an orientation reversing map, or has degree* 0*, and* $f$ *does not have a periodic orbits with even index of period a power of* 2 *different from* 1*, then* $f$ *has infinitely many periodic points.*

*Proof.* If the map $f$ is orientation preserving map on $\mathbb{S}^{2n+1}$ then $\zeta_f(t) = 1$. If $f$ has finitely many periodic points, according to Theorem 1, we have:

$$\prod_j(1 \pm t^{q_j}) = \prod_i(1 \pm t^{p_i}). \tag{8}$$

We take the special factorization. If $f$ has periodic points of index $u$ with a odd period $p$. Then the factor $1 \pm t^p$ is on one side of the the equation (8), say on the right hand side, i.e. in this case $u$ is odd. Since there are no periodic points of even index with periods of the form $kp$ with $k$ even and 1, so there are no factors of the form $1 \pm t^{kp}$ on the left hand side of (8). Therefore this equality does not hold, so the map $f$ has infinitely many periodic points. This completes the proof of statement (a).

If the index of all periodic points of period of power of 2 and fixed points of the map $f$, have the same parity, then in (8) all the factors with a power of 2 and $1 \pm t$ are only on one side of the equation. So the equality (8) is not possible. Hence the map $f$ has infinitely many periodic points. This completes the proof of statement (b).

If $f$ is an orientation reversing map then $\zeta_f(t) = (1+t)(1-t)^{-1}$. So $f$ has finitely many periodic points, we have the equation:

$$(1+t)\prod(1 \pm t^{q_j}) = (1-t)\prod(1 \pm t^{p_i}). \tag{9}$$

Since $f$ does not have periodic points of even index with period a power of 2, nor does have fixed points with even index. There is no factors of the form $1-t$ on the left hand side of (9). On the right hand side of (9) there is at least one factor of this form. Therefore the equality of (9) does not hold, so $f$ has infinitely many periodic points. If the degree of $f$ is 0, then $\zeta_f(t) = (1-t)^{-1}$ and a similar argument allow us to conclude that $f$ has infinitely many periodic points . This completes the proof of statement (c). $\square$

**Theorem 7.** *Let $X$ be a closed manifold and $f : X \to X$ be a $C^1$ map with all its periodic points hyperbolic such that $\zeta_f(t) = 1$.*

(a) *If $f$ has a periodic point with an odd period $p$, with index $u$ and it does not have periodic points of periods a multiple of $p$ with index $v \not\equiv u \pmod 2$. Then $f$ has infinitely many periodic points.*
(b) *Assume that $f$ has periodic points of period a power of 2 whose indexes have the same parity. Let $u$ be this parity. If $f$ does not have fixed points of index $v$ with $v \not\equiv u \pmod 2$. Then $f$ has infinitely many periodic points.*

*Proof.* It is the same proof than in the statements (a) and (b) of Theorem 6. $\square$

The possible zeta functions for quasi–unipotent maps on closed orientable surfaces $M_g$ are computed in [6] for $0 \leq g \leq 3$, where $g$ is the genus of the surface. For non–orientable surfaces $N_g$ the zeta functions are calculated in [7] for $1 \leq g \leq 9$. These Lefschetz zeta functions are of the type considered in the previous theorems. Similarly for the quasi–unipotent maps on the 3 and 4 dimensional torus studied in [5].

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## References

[1] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[2] N.M. Dos Santos and R. Urzúa, Minimal homeomorphisms on low–dimensional tori, *Ergod. Th. and Dynam. Sys.* **29** (2009), 1515–1528.

[3] J. Franks, Some smooth maps with infinitely many hyperbolic points. *Trans. Amer. Math. Soc.* **226** (1977), 175–179,

[4] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.

[5] J.L.G. Guirao and J. Llibre Minimal Lefschetz sets of periods for Morse–Smale diffeomorphisms on the $n$–dimensional torus, *J. Difference Equations and Applications* **16** (2010), 689–703.

[6] J. Llibre and V.F. Sirvent, Minimal sets of periods for Morse–Smale diffeomorphisms on orientable compact surfaces, *Houston J. of Math.* **35** (2009), 835–855.

[7] J. Llibre and V.F. Sirvent, Minimal sets of periods for Morse–Smale diffeomorphisms on non–orientable compact surfaces without boundary, *J. Difference Equations and Applications* **19** (2013), 402–417

[8] M. Shub and D. Sullivan, Homology theory and dynamical systems, *Topology* **14** (1975), 109–132.

[9] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

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